Cambridge Assessment International Education

Cambridge International A Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **October/November 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level components, and some Cambridge O Level components.

This document consists of **21** printed pages.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1. Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2. Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3. Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5. Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6. Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

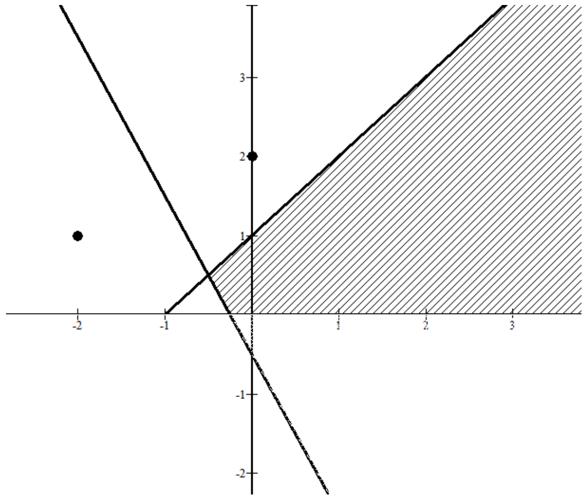
FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	Use correct quotient or product rule	*M1	
	Obtain correct derivative in any form	A1	e.g. $\frac{(1-3x)2x-x^2(-3)}{(1-3x)^2}\left(=\frac{2x-3x^2}{(1-3x)^2}\right)$ or $3x^2(1-3x)^{-2}+2x(1-3x)^{-1}$.
	Equate derivative to 8 and solve for x	DM1	$75x^2-50x+8=(15x-4)(5x-2)$.
	Obtain answers $x=\frac{2}{5}$ and $\frac{4}{15}$	A1	Exact values required.
	Obtain answers $y=-\frac{4}{5}$ and $\frac{16}{45}$	A1	Allow A1 for one correct point.
		5	

Question	Answer	Marks	Guidance
2	Show points representing 2i and –2 + i	**B1**	Can be implied if the correct perpendicular is drawn.
	Show perpendicular bisector of *their* (2i and –2 + i)	**B1FT**	
	Show correct half–line of gradient 1 from point (–1, 0)	**B1**	Should pass through (0, 1).
	Correct loci and shade correct region	**B1**	
		4	

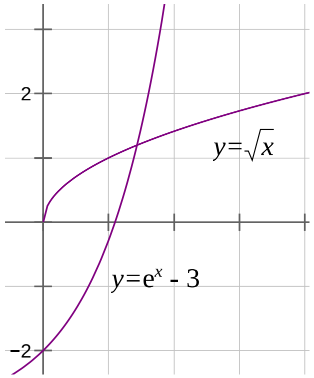
Question	Answer	Marks	Guidance
3	State or imply that $\ln y = \ln a + x \ln b$	**B1**	
	Carry out a completely correct method for finding $\ln a$ or $\ln b$	**M1**	$3.7 = \ln a + \ln b$ and $6.46 = \ln a + 2.2\ln b$ leading to $\ln a = 1.4$, $\ln b = 2.3$.
	Obtain value $a = 4.06$	**A1**	
	Obtain value $b = 9.97$	**A1**	SC B1 for $a = e^{1.4}$ and $b = e^{2.3}$.
	Alternative Method for Question 3		
	$e^{3.7} = ab^1$ and $e^{6.46} = ab^{2.2}$	**B1**	
	Divide to obtain $e^{2.76} = b^{1.2}$ and state or imply $2.76 = 1.2\ln b$	**M1**	
	Obtain value $a = 4.06$	**A1**	
	Obtain value $b = 9.97$	**A1**	
		4	

Question	Answer	Marks	Guidance
4(a)	Multiply numerator and denominator by $a + 5\mathrm{i}$	**M1**	OE
	Use $\mathrm{i}^2 = -1$	**M1**	At least once.
	Obtain answer $\dfrac{3a-10}{a^2+25}+\dfrac{2a+15}{a^2+25}\mathrm{i}$	**A1**	
	Alternative Method for Question 4(a)		
	Multiply $x + \mathrm{i}y$ by $a - 5\mathrm{i}$ and use $\mathrm{i}^2 = -1$	**M1**	
	Compare real and imaginary parts	**M1**	$3 = ax + 5y$, $2 = ay - 5x$.
	Obtain answer $\dfrac{3a-10}{a^2+25}+\dfrac{2a+15}{a^2+25}\mathrm{i}$	**A1**	
		3	
4(b)	State or imply Im**(a)** ÷ Re**(a)** = 1	**M1**	Or Im**(a)** = Re**(a)** or equivalent for *their* u.
	Obtain answer $a = 25$	**A1**	
		2	

Question	Answer	Marks	Guidance
5(a)	Use correct trig formulae and obtain an equation in sin *x* and cos *x*	***M1**	Allow one sign error.
	Obtain a correct equation in any form	**A1**	e.g $2\cos x\sin\frac{\pi}{6} = -2\sin x\sin\frac{\pi}{3}$.
	Substitute exact trig ratios and obtain an expression for tan *x*	**DM1**	Allow one sign error.
	Obtain answer $\tan x = -\dfrac{1}{\sqrt{3}}$	**A1**	Or exact equivalent.
		4	
5(b)	Obtain answer, e.g. $x = \dfrac{5\pi}{6}$	**B1**	
	Obtain second answer, e.g. $x = \dfrac{11\pi}{6}$ and no others in the interval	**B1FT**	FT first answer + π (provided 0 ⩽ first answer ⩽ π). Or FT first answer – π (provided π ⩽ first answer ⩽ 2π). Ignore any answers outside interval.
		2	

Question	Answer	Marks	Guidance
6(a)	State correct derivative of x **or** y with respect to t	**B1**	$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$, $\frac{dy}{dt} = \frac{1}{t}$.
	Use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$	**M1**	Use correct chain rule.
	Obtain answer $\frac{dy}{dx} = \frac{2}{\sqrt{t}}$	**A1**	Or simplified equivalent e.g $2t^{\frac{-1}{2}}$ or $\frac{2\sqrt{t}}{t}$.
		3	
6(b)	State or imply *their* $\frac{dy}{dx} = \frac{1}{2}$	**M1**	
	Obtain $\sqrt{t} = 4$	**A1**	Or equivalent.
	Obtain answer (7, ln 16)	**A1**	Or exact equivalent. Can state the two components separately.
		3	

Question	Answer	Marks	Guidance
7	Separate variables correctly	**B1**	$\int \tan\theta \, d\theta = \int \frac{x}{x^2+3}\, dx$. Condone missing integral signs **or** missing dx, dθ. Can be implied by later work.
	Obtain term –ln(cos θ)	**B1**	Or equivalent e.g. ln(sec θ).
	Obtain term of the form $a\ln\left(x^2+3\right)$	**M1**	
	Obtain term $\frac{1}{2}\ln\left(x^2+3\right)$	**A1**	
	Use x = 1, θ = 0 to evaluate a constant or as limits in a solution containing terms of the form $a\ln\left(x^2+3\right)$ and $b\ln(\cos\theta)$	**M1**	If they have rearranged then the constant must be of the correct form.
	Obtain correct answer in any form	**A1**	$\frac{1}{2}\ln\left(x^2+3\right) = -\ln\cos\theta + \ln 2$.
	Obtain final answer $x^2 = \frac{4}{\cos^2\theta} - 3$	**A1**	Or equivalent e.g. $x^2 = 4\sec^2 x - 3$. lns removed.
		7	

Question	Answer	Marks	Guidance
8(a)	Sketch a relevant graph, e.g. $y = e^x - 3$ $y = e^x - 3$ Should cut vertical axis at (0, –2) and have increasing gradient.	**B1**	
	Sketch a second relevant graph, e.g. $y = \sqrt{x}$ and justify the given statement $y = \sqrt{x}$ should start at (0, 0) and have reducing grading	**B1**	Ignore anything outside 1st and 4th quadrants. For second B1 need to mark intersection with a dot, a cross, or say root at point of intersection, or equivalent.
		2	
8(b)	Calculate the values of a relevant expression or pair of expressions at $x = 1$ and $x = 2$	**M1**	
	Complete the argument correctly with correct calculated values	**A1**	e.g. $1 > -0.28..$, $1.41 < 4.39..$ $1.28 > 0$, $-2.98 < 0$.
		2	
8(c)	State $x = \ln\left(3+\sqrt{x}\right)$ and rearrange to the given equation $\sqrt{x} = e^x - 3$	**B1**	Or rearrange $\sqrt{x} = e^x - 3$ to $x = \ln(3+\sqrt{x})$ and state iterative formula of $x_{n+1} = \ln(3+\sqrt{x_n})$. AG
		1	

Question	Answer	Marks	Guidance
8(d)	Use the iterative process correctly at least once	**M1**	
	Obtain final answer 1.43	**A1**	
	Show sufficient iterations to at least 4 d.p. to justify 1.43 to 2 d.p. or show there is a sign change in the interval (1.425, 1.435) Condone recovery and small differences in the final figure in the iteration	**A1**	e.g. 1, 1.3864, 1.4297, 1.4341, … 1.5, 1.4210, 1.4332, 14.344, 1.4345, … 2, 1.4848, 1.4395, 1.4350, 1.4346, 1.4345, …
		3	

Question	Answer	Marks	Guidance
9(a)	Use the correct product rule	***M1**	Condone error in chain rule.
	Obtain correct derivative in any form	**A1**	e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2}{2}\mathrm{e}^{-\frac{x^2}{4}} + \mathrm{e}^{-\frac{x^2}{4}}$.
	Equate derivative to zero and solve for x	**DM1**	
	Obtain answer $\left(\sqrt{2}, \sqrt{2}\mathrm{e}^{-\frac{1}{2}}\right)$	**A1**	Or exact equivalent. Can state the components separately.
		4	

Question	Answer	Marks	Guidance
9(b)	State or imply $dx = \frac{1}{2}u^{-\frac{1}{2}}du$	**B1**	Or equivalent e.g. $du = 2xdx$. Alternative substitution: $u = -\frac{1}{4}x^2$.
	Substitute for x and dx	**M1**	
	Obtain correct integral $\frac{1}{2}\int e^{-\frac{1}{4}u}\,du$	**A1**	OE
	Use correct limits in an integral of the form $ae^{-\frac{1}{4}u}$ or $ae^{-\frac{1}{4}x^2}$	**M1**	$u = 9$ and $u = 0$ or $x = 3$ and $x = 0$.
	Obtain answer $2 - 2e^{-\frac{9}{4}}$	**A1**	Or exact equivalent.
	Alternative Method for Question 9(b)		
	$\int xe^{-\frac{1}{4}x^2}dx = ae^{-\frac{1}{4}x^2}$	**M1**	Recognition used.
	a negative	**A1**	
	$a = -2$	**A1**	
	Use correct limits in an integral of the form $ae^{-\frac{1}{4}x^2}$	**M1**	$x = 3$ and $x = 0$.
	Obtain answer $2 - 2e^{-\frac{9}{4}}$	**A1**	Or exact equivalent.
		5	

Question	Answer	Marks	Guidance
10(a)	State or imply the form $\dfrac{A}{1-2x}+\dfrac{B}{2+x}+\dfrac{C}{(2+x)^2}$	**B1**	
	Use a correct method for finding a coefficient	**M1**	$A(2 + x)^2 + B(1 – 2x)(2 + x) + C(1 – 2x)$ $= 24x + 13$.
	Obtain one of $A = 4$, $B = 2$ and $C = –7$	**A1**	If errors in equating still allow A marks for A and C.
	Obtain a second value	**A1**	
	Obtain the third value	**A1**	Mark the form $\dfrac{A}{1-2x} + \dfrac{Dx+E}{(2+x)^2}$, where $A = 4$, $D = 2$ and $E = –3$, B1 M1 A1 A1 A1 as above. If there are extra term in partial fractions, that is 4 unknowns A, B, D and E then B0 unless recover at end, e.g. by setting $B = 0$. If B set to any value other than 0 and all coefficients correctly found to their new values then allow all A marks, but still B0 for partial fraction expression. Hence A1 for each coefficient, but nothing for coefficient set to specific value. Another case of extra term in partial fraction expression, namely $+ F$, mark as above but need $F = 0$ to recover B1.
		5	

Question	Answer	Marks	Guidance
10(b)	Use a correct method to find the first two terms of the expansion of $(1-2x)^{-1}$, $(2+x)^{-1}$, $(2+x)^{-2}$, $\left(1+\frac{x}{2}\right)^{-1}$ or $\left(1+\frac{x}{2}\right)^{-2}$	M1	Symbolic coefficients are not sufficient for the M1.
	Obtain correct un-simplified expansions up to the term in x^2 of each partial fraction	A1 FT	$A\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2}(-2x)^2+..\right)$ $A = 4$.
		A1 FT	$\frac{B}{2}\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2}\left(\frac{x}{2}\right)^2+...\right)$ $B = 2$.
		A1 FT	$\frac{C}{4}\left(1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2}\left(\frac{x}{2}\right)^2+...\right)$ $C = -7$ $= 4(1 + 2x + 4x^2) + 2/2(1 - x/2 + x^2/4) - 7/4(1 - x + 3x^2/4)$ $= (4 + 1 - 7/4) + (8 - 1/2 + 7/4)\,x + (16 + 1/4 - 21/16)\,x^2$ The FT is on A, B, C.

Question	Answer	Marks	Guidance
	Obtain final answer $\frac{13}{4}+\frac{37}{4}x+\frac{239}{16}x^2$	A1	OE $(Dx + E)/4\ [1 + (-2)(x/2) + (-2)(-3)(x/2)^2/2 \ldots]$ $D = 2 \quad E = -3$ The FT is on A, D, E. Maclaurin's Series f(0) = 13/4 B1 f'(0) = 37/4 B1 f''(0) = 239/8 B1. $\frac{13}{4}+\frac{37}{4}x+\frac{239}{8}x^2/2$ or equivalent M1 A1. If $1+\frac{37}{4}x+\frac{239}{8}x^2/2$ then M0 A0 unless *their* f(0) actually is 1. For the A, D, E form of fractions, give M1 A1FT A1FT for the expanded partial fractions, then, if $D \neq 0$, M1 for multiplying out fully, and A1 for the final answer. If final answer has been multiplied throughout (e.g. by 16) then A0 at the end
		5	
10(c)	$\lvert x\rvert < \frac{1}{2}$	B1	OE
		1	

Question	Answer	Marks	Guidance
11(a)	Obtain 3**i** + 2**j** + **k**	B1	Accept coordinates in place of position vector.
		1	

Question	Answer	Marks	Guidance
11(b)	$\overrightarrow{AM}$ or $\overrightarrow{AP}$ correct soi	B1	$\overrightarrow{AM} = 2\mathbf{j} + \mathbf{k}$, or $\overrightarrow{AP} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
	Carry out correct process for evaluating the scalar product of $\overrightarrow{AM}$ and $\overrightarrow{AP}$	M1	or $\overrightarrow{MA}$ and $\overrightarrow{PA}$: $0+2+2$.
	Using the correct process for the moduli, divide the scalar product by the product of the moduli and obtain the inverse cosine of the result	M1	For their vectors. $\theta = \cos^{-1}\left(\frac{4}{3\sqrt{5}}\right)$.
	Obtain answer 53.4° or 0.932^c	A1	
		4	
11(c)	Find $\overrightarrow{PQ}$ (or $\overrightarrow{QP}$) for a general point Q on the line passing through O and M,	B1 FT	e.g. $\boldsymbol{PQ} = -(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Follow *their M*.
	Calculate the scalar product of $\overrightarrow{PQ}$ and a direction vector for the line passing through O and M and equate to zero	*M1	
	Solve and obtain correct solution e.g. $\mu = -\frac{1}{2}$	A1	
	Carry out method to calculate PQ	DM1	$\sqrt{.5^2 + 0 + 1.5^2}$.
	Obtain answer $\frac{\sqrt{10}}{2}$	A1	Or exact equivalent.
	Alternative Method 1 for Question 11(c)		
	Find $\overrightarrow{PQ}$ (or $\overrightarrow{QP}$) for a general point Q on the line passing through O and M,	B1 FT	e.g. $\boldsymbol{PQ} = -(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Follow *their M*.
	Use a correct method to express PQ^2 (or PQ) in terms of μ	*M1	
	Obtain a correct equation in any form	A1	e.g. $PQ^2 = (1 + 3\mu)^2 + (1 + 2\mu)^2 + (2 + \mu)^2$

Question	Answer	Marks	Guidance
11(c)	Carry out a complete method for finding its minimum	**DM1**	e.g. $6(1+3\mu)+4(1+2\mu)+2(2+\mu)=0,\ \mu=-\frac{1}{2}$.
	Obtain answer $\frac{\sqrt{10}}{2}$	**A1**	Or exact equivalent.
	Alternative Method 2 for Question 11(c)		
	Calling (0, 0, 0) A, state $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) in component form, e.g. **i** + **j** + 2**k**	**B1**	
	Use a scalar product to find the projection of $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) on the line passing through O and M	**M1**	
	Obtain correct answer $\frac{7}{\sqrt{14}}$	**A1**	OE
	Use Pythagoras to find the perpendicular	**M1**	$d=\sqrt{AP^2-AQ^2}=\sqrt{1+1+2^2-\left(\frac{7}{\sqrt{14}}\right)^2}$.
	Obtain answer $\frac{\sqrt{10}}{2}$	**A1**	Or exact equivalent.
	Alternative Method 3 for Question 11(c)		
	Calling (0, 0, 0) A, state $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) in component form, e.g. **i** + **j** + 2**k**	**B1**	
	Calculate the vector product of $\overrightarrow{PA}$ and a direction vector for the line passing through O and M	**M1**	
	Obtain correct answer, e.g. 3**i** – 5**j** + **k**	**A1**	

Question	Answer	Marks	Guidance
11(c)	Divide modulus of the product by that of the direction vector	**M1**	e.g. $\frac{\sqrt{3^2+5^2+1^2}}{\sqrt{3^2+2^2+1^2}}$.
	Obtain answer $\frac{\sqrt{10}}{2}$	**A1**	Or exact equivalent.
		5	